2018

AP® CollegeBoard

AP Calculus AB

Sample Student Responses and Scoring Commentary

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Question 1

(a) $\int_0^{300} r(t)\,dt = 270$ According to the model, 270 people enter the line for the escalator during the time interval $0 \le t \le 300$.	$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$
(b) $20 + \int_0^{300} (r(t) - 0.7)\,dt = 20 + \int_0^{300} r(t)\,dt - 0.7 \cdot 300 = 80$ According to the model, 80 people are in line at time $t = 300$.	$2 : \begin{cases} 1 : \text{considers rate out} \\ 1 : \text{answer} \end{cases}$
(c) Based on part (b), the number of people in line at time $t = 300$ is 80. The first time t that there are no people in line is $300 + \frac{80}{0.7} = 414.286$ (or 414.285) seconds.	1 : answer
(d) The total number of people in line at time t, $0 \le t \le 300$, is modeled by $20 + \int_0^t r(x)\,dx - 0.7t$. $r(t) - 0.7 = 0 \Rightarrow t_1 = 33.013298, t_2 = 166.574719$	$4 : \begin{cases} 1 : \text{considers } r(t) - 0.7 = 0 \\ 1 : \text{identifies } t = 33.013 \\ 1 : \text{answers} \\ 1 : \text{justification} \end{cases}$

t	People in line for escalator
0	20
t_1	3.803
t_2	158.070
300	80

The number of people in line is a minimum at time $t = 33.013$ seconds, when there are 4 people in line.

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1. People enter a line for an escalator at a rate modeled by the function r given by q

$$r(t) = \begin{cases} 44\left(\frac{t}{100}\right)^3\left(1 - \frac{t}{300}\right)^7 & \text{for } 0 \le t \le 300 \\ 0 & \text{for } t > 300, \end{cases}$$

where $r(t)$ is measured in people per second and t is measured in seconds. As people get on the escalator, they exit the line at a constant rate of 0.7 person per second. There are 20 people in line at time $t = 0$.

(a) How many people enter the line for the escalator during the time interval $0 \le t \le 300$?

$$\int_0^{300} 44\left(\frac{t}{100}\right)^3\left(1-\frac{t}{300}\right)^7 dt = \boxed{270}$$

(b) qDuring the time interval $0 \le t \le 300$, there are always people in line for the escalator. How many people are in line at time $t = 300$?

$.7(300) = 210$

$20 + 270 = 290$

$290 - 210 = \boxed{80}$

Continue question 1 on page 5.

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(c) For $t > 300$, what is the first time t that there are no people in line for the escalator?

$(t-300)(.7) - 80 = 0$

$.7t - 210 - 80 = 0$

$.7t = +290$

$\boxed{t = 414.286s}$

(d) JFor $0 \le t \le 300$, at what time t is the number of people in line a minimum? To the nearest whole number, find the number of people in line at this time. Justify your answer.

p= total people

$\frac{dp}{dt} = r(t) - .7$

$0 = r(t) - .7$

$t = 166.575$

$t = 33.013$

$p(t) = \int_0^t r(\cancel{t}) - .7 \, dx + 20$

t	p(t)
0	20
33.013	3.803
166.575	158.07014
300	80

minimum at time t= 33.013 s when 4 people are in line

1B
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1. People enter a line for an escalator at a rate modeled by the function r given by

$$r(t) = \begin{cases} 44\left(\frac{t}{100}\right)^3\left(1 - \frac{t}{300}\right)^7 & \text{for } 0 \le t \le 300 \\ 0 & \text{for } t > 300, \end{cases}$$

$S(0) = 20$

where $r(t)$ is measured in people per second and t is measured in seconds. As people get on the escalator, they exit the line at a constant rate of 0.7 person per second. There are 20 people in line at time $t = 0$.

(a) How many people enter the line for the escalator during the time interval $0 \le t \le 300$? v

$\int_0^{300} 44\left(\frac{t}{100}\right)^3\left(1-\frac{t}{300}\right)^7 dt =$ 270 people enter the line for the escalator during the time interval $0 \le t \le 300$

(b) During the time interval $0 \le t \le 300$, there are always people in line for the escalator. How many people are in line at time $t = 300$?

$L(t) = 0.7$

$20 + \int_0^{300} r(t)\,dt - \int_0^{300} 0.7\,dt$

$20 + 270 - 210$

At time $t = 300$ seconds, there are 80 people in line for the escalator.

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Continue question 1 on page 5.

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(c) For $t > 300$, what is the first time t that there are no people in line for the escalator?

For $t > 300$, the first time t that there are no people in line for the escalator is $t = 325$

(d) For $0 \le t \le 300$, at what time t is the number of people in line a minimum? To the nearest whole number, find the number of people in line at this time. Justify your answer.

$44\left(\frac{t}{100}\right)^3\left(1-\frac{t}{300}\right)^7 - 0.7$

$44\left(\frac{t}{100}\right)^3\left(1-\frac{t}{300}\right)^7 = 0.7$

$20 + \int_0^{33.013298 \rightarrow A} r(t)dt - \int_0^A 0.7dt$

At time $t = 33.013$ seconds there is a minimam of 3.803 people on the escalator.

1C
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1. People enter a line for an escalator at a rate modeled by the function r given by

$$r(t) = \begin{cases} 44\left(\frac{t}{100}\right)^3\left(1 - \frac{t}{300}\right)^7 & \text{for } 0 \le t \le 300 \\ 0 & \text{for } t > 300, \end{cases}$$

where $r(t)$ is measured in people per second and t is measured in seconds. As people get on the escalator, they exit the line at a constant rate of 0.7 person per second. There are 20 people in line at time $t = 0$.

(a) How many people enter the line for the escalator during the time interval $0 \le t \le 300$?

$r(t) = 44\left(\frac{t}{100}\right)^3\left(1-\frac{t}{300}\right)^7 \quad (0,300)$

$\text{people} = \int_0^{300} 44\left(\frac{t}{100}\right)^3\left(1-\frac{t}{300}\right)^7 dt$

$\text{people} = \boxed{56700 \text{ people}}$

(b) During the time interval $0 \le t \le 300$, there are always people in line for the escalator. How many people are in line at time $t = 300$?

$20 + \int_0^{300} r(t)\, dt - \left(\int_0^{300} (0.7\, dt)\right)$

$\left[20 + \int_0^{300} 44\left(\frac{t}{100}\right)^3\left(1-\frac{t}{300}\right)^7 dt\right] - \int_0^{300} .7\, dt$

$= \boxed{80 \text{ people}}$

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(c) For $t > 300$, what is the first time t that there are no people in line for the escalator?

$r(t) = 0 \qquad t > 300$

There are no people in line for the escalator first at time $t = 300$.

(d) For $0 \leq t \leq 300$, at what time t is the number of people in line a minimum? To the nearest whole number, find the number of people in line at this time. Justify your answer.

$\underset{\text{enter}}{\text{rate}} - \underset{\text{exit}}{\text{rate}} = 0$

$* \, r(t) = 0$ and changes inc → dec

$r(t) = 44\left(\frac{t}{100}\right)^3\left(1 - \frac{t}{300}\right)^7 = 0$

$t = 150$ sec.

Question 1

Overview

The context of this problem is a line of people waiting to get on an escalator. The function r models the rate at which people enter the line, where $r(t) = 44\left(\frac{t}{100}\right)^3\left(1 - \frac{t}{300}\right)^7$ for $0 \le t \le 300,$ and $r(t) = 0$ for $t > 300;$ $r(t)$ is measured in people per second, and t is measured in seconds. Further, it is given that people exit the line to get on the escalator at a constant rate of 0.7 person per second and that there are 20 people in the line at time $t = 0.$ In part (a) students were asked how many people enter the line for the escalator during the time interval $0 \le t \le 300.$ A correct response demonstrates the understanding that the number of people entering the line during this time interval is obtained by integrating the rate at which people enter the line across the time interval. Thus, this number is the value of the definite integral $\int_0^{300} r(t)\, dt.$ A numerical value for this integral should be obtained using a graphing calculator. In part (b) students were given that there are always people in line during the time interval $0 \le t \le 300$ and were asked to determine the number of people in line at time $t = 300.$ A correct response should take into account the 20 people in line initially, the number that entered the line as determined in part (a), and the number of people that exit the line to get on the escalator. It was given in the problem statement that people exit the line at a constant rate of 0.7 person per second, so the number of people that exit the line to get on the escalator can be found by multiplying this constant rate times the duration of the interval, namely 300 seconds. In part (c) students were asked for the first time t beyond $t = 300$ when there are no people in line for the escalator. Because no more people join the line after $t = 300$ seconds, and people exit the line at the constant rate of 0.7 person per second, dividing the answer to part (b) by 0.7 gives the number of seconds beyond $t = 300$ before the line empties for the first time. Adding this quotient to 300 produces the answer. In part (d) students were asked when, during the time interval $0 \le t \le 300,$ is the number of people in line a minimum, and to determine the number of people in line (to the nearest whole number) at that time, with the added admonition to justify their answer. The Extreme Value Theorem guarantees that the number of people in line at time $t,$ given by the expression $20 + \int_0^t r(x)\, dx - 0.7t,$ attains a minimum on the interval $0 \le t \le 300.$ Correct responses should demonstrate that the rate of change of the number of people in line is given by $r(t) - 0.7.$ Solving for $r(t) - 0.7 = 0$ within the interval $0 < t < 300$ yields two critical points, t_1 and $t_2,$ so candidates for the time when the line is a minimum are $t = 0,$ $t_1,$ $t_2,$ and $t = 300.$ The number of people in line at times t_1 and t_2 is computed from $20 + \int_0^{t_1} r(x)\, dx - 0.7t_1$ and $20 + \int_0^{t_2} r(x)\, dx - 0.7t_2.$ The answer is the least of 20, these two computed values (to the nearest whole number), and the answer to part (b), together with the corresponding time t for this minimum value.

For part (a) see LO 3.3B(b)/EK 3.3B2, LO 3.4A/EK 3.4A2, LO 3.4E/EK 3.4E1. For parts (b) and (c), see LO 3.4A/EK 3.4A2, LO 3.4E/EK 3.4E1. For part (d) see LO 1.2B/EK 1.2B1, LO 2.3C/EK 2.3C3, LO 3.3B(b)/EK 3.3B2, LO 3.4A/EK 3.4A2, LO 3.4E/EK 3.4E1. This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, building notational fluency, and communicating.

Sample: 1A
Score: 9

The response earned all 9 points: 2 points in part (a), 2 points in part (b), 1 point in part (c), and 4 points in part (d). In part (a) the response earned the first point for $\int_0^{300} 44\left(\frac{t}{100}\right)^3\left(1 - \frac{t}{300}\right)^7 dt.$ The response earned the

Question 1 (continued)

second point for the answer 270. In part (b) the response earned the first point for -210 on the right side of the response area. The response earned the second point for the answer 80. In part (c) the response earned the point for the answer 414.286. What appears to be a fourth digit of 5 is not a digit but the letter s for seconds. Units are not required to earn any points in this question. In part (d) the response earned the first point for $0 = r(t) - .7$ in line 2. The response earned the second point with $t = 33.013$ in line 4. The response earned the third point for the boxed information. What appears to be a fourth digit of 5 is not a digit but the letter s for seconds. The response earned the fourth point for the candidates test demonstrated with the table. The expression for the function p, identified as "total people," supports how the values at 33.013 and 166.575 are produced.

Sample: 1B
Score: 6

The response earned 6 points: 2 points in part (a), 2 points in part (b), no point in part (c), and 2 points in part (d). In part (a) the response earned the first point for $\int_0^{300} 44\left(\frac{t}{100}\right)^3\left(1-\frac{t}{300}\right)^7 dt.$ The response earned the second point for the answer 270. In part (b) the response earned the first point with the term $-\int_0^{300} 0.7\, dt.$ The response earned the second point for the answer 80. In part (c) the response did not earn the point because what is presented is incorrect. In part (d) the response earned the first point in line 2 with the equation $44\left(\frac{t}{100}\right)^3\left(1-\frac{t}{300}\right)^7 = 0.7.$ The response earned the second point with $t = 33.013$ in line 3. The response did not earn the third point for the answers because 3.803 is not rounded to a whole number. The response did not earn the fourth point because it does not have a complete justification.

Sample: 1C
Score: 3

The response earned 3 points: 1 point in part (a), 2 points in part (b), no point in part (c), and no points in part (d). In part (a) the response earned the first point for $\int_0^{300} 44\left(\frac{t}{100}\right)^3\left(1-\frac{t}{300}\right)^7 dt.$ The response did not earn the second point because 56700 is incorrect. In part (b) the response earned the first point with the term $-\left(\int_0^{300}(0.7\, dt)\right)$ in line 1. The response earned the second point for the answer 80. In part (c) the response did not earn the point because what is presented is incorrect. In part (d) the response did not earn the first point with any of the equations presented. The equation at the top right "rate enter $-$ rate exit $= 0$" is too formulaic and not specific to the question. The response does not identify $t = 33.013$ and did not earn the second point. As a result, the response is not eligible to earn the remaining 2 points.

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Sample Student Responses and Scoring Commentary

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Question 2

(a) $v'(3) = -2.118$ The acceleration of the particle at time $t = 3$ is -2.118.	1 : answer
(b) $x(3) = x(0) + \int_0^3 v(t)\,dt = -5 + \int_0^3 v(t)\,dt = -1.760213$ The position of the particle at time $t = 3$ is -1.760.	$3 : \begin{cases} 1 : \int_0^3 v(t)\,dt \\ 1 : \text{uses initial condition} \\ 1 : \text{answer} \end{cases}$
(c) $\int_0^{3.5} v(t)\,dt = 2.844$ (or 2.843) $\int_0^{3.5} \lvert v(t) \rvert\,dt = 3.737$ The integral $\int_0^{3.5} v(t)\,dt$ is the displacement of the particle over the time interval $0 \le t \le 3.5$. The integral $\int_0^{3.5} \lvert v(t) \rvert\,dt$ is the total distance traveled by the particle over the time interval $0 \le t \le 3.5$.	$3 : \begin{cases} 1 : \text{answers} \\ 2 : \text{interpretations of } \int_0^{3.5} v(t)\,dt \\ \quad \text{and } \int_0^{3.5} \lvert v(t) \rvert\,dt \end{cases}$
(d) $v(t) = x_2'(t)$ $v(t) = 2t - 1 \Rightarrow t = 1.57054$ The two particles are moving with the same velocity at time $t = 1.571$ (or 1.570).	$2 : \begin{cases} 1 : \text{sets } v(t) = x_2'(t) \\ 1 : \text{answer} \end{cases}$

2. A particle moves along the x-axis with velocity given by $v(t) = \frac{10\sin(0.4t^2)}{t^2 - t + 3}$ for time $0 \le t \le 3.5$.

The particle is at position $x = -5$ at time $t = 0$.

(a) Find the acceleration of the particle at time $t = 3$.

$v'(3) = -2.1182$

(b) Find the position of the particle at time $t = 3$. U

$-5 + \int_0^3 v(t)\,dt = -1.7602$

Continue question 2 on page 7.

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(c) Evaluate $\int_0^{3.5} v(t)\,dt$, and evaluate $\int_0^{3.5} |v(t)|\,dt$. Interpret the meaning of each integral in the context of the problem.

$\int_0^{3.5} v(t)\,dt = 2.8439$, which is the displacement of the particle from $t=0$ to $t=3.5$

$\int_0^{3.5} |v(t)|\,dt = 3.7371$, which is the total distanse the particle traveled from $t=0$ to $t=3.5$

(d) A second particle moves along the x-axis with position given by $x_2(t) = t^2 - t$ for $0 \le t \le 3.5$. At what time t are the two particles moving with the same velocity?

$v(t) = 2t-1$

$t = 1.5705$

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2B
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2. A particle moves along the x-axis with velocity given by $v(t) = \dfrac{10\sin(0.4t^2)}{t^2 - t + 3}$ for time $0 \le t \le 3.5$.

The particle is at position $x = -5$ at time $t = 0$.

(a) Find the acceleration of the particle at time $t = 3$.

$v(t) = \dfrac{10\sin(0.4t^2)}{t^2 - t + 3}$

$v'(3) = -2.11819$

(b) Find the position of the particle at time $t = 3$.

$-5 + \int_0^3 \dfrac{10\sin(0.4t^2)}{t^2 - t + 3}\,dt = -1.760$

Continue question 2 on page 7.

2B

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(c) Evaluate $\int_0^{3.5} v(t)\, dt$, and evaluate $\int_0^{3.5} |v(t)|\, dt$. Interpret the meaning of each integral in the context of the problem.

$\int_0^{3.5} v(t)\, dt = 2.843$

$\int_0^{3.5} v(t)$ means the position of the particle at time $t = 3.5$ which is 2.843 units.

$\int_0^{3.5} |v(t)| = 3.737$

$\int_0^{3.5} |v(t)|$ means the total distance that the particle has traveled from $t=0$ to $t=3.5$

(d) A second particle moves along the x-axis with position given by $x_2(t) = t^2 - t$ for $0 \le t \le 3.5$. At what time t are the two particles moving with the same velocity?

$x_2(t) = t^2 - t$

$v_2(t) = 2t \; dt$

There is not a time where the two particles are moving with the same velocity because their velocity functions never intersect with each other.

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2C
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2. A particle moves along the x-axis with velocity given by $v(t) = \dfrac{10\sin\left(0.4t^2\right)}{t^2 - t + 3}$ for time $0 \le t \le 3.5$.

The particle is at position $x = -5$ at time $t = 0$.

(a) Find the acceleration of the particle at time $t = 3$.

$$a(t) = v'(t) = \frac{(2t-1)(10\sin(0.4t^2)) - 0.8t(t^2-t+3)(10\cos(0.4t^2))}{(t^2-t+3)^2}$$

$$a(3) = \frac{(6-1)(10\sin(3.6)) - 2.4(9-3+3)(10\cos(3.6))}{(9-3+3)^2}$$

$$= \frac{50\sin(3.6) - 216\cos(3.6)}{81}$$

$$= 2.118$$

(b) Find the position of the particle at time $t = 3$. [

$$P = \int_0^3 v(t)\,dt = \int_0^3 \frac{10\sin(0.4t^2)}{t^2-t+3}\,dt = 3.240 \text{ m}$$

Continue question 2 on page 7.

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(c) Evaluate $\int_0^{3.5} v(t)\,dt$, and evaluate $\int_0^{3.5} |v(t)|\,dt$. Interpret the meaning of each integral in the context of the problem.

$\int_0^{3.5} v(t)dt = \int_0^{3.5} \frac{10\sin(0.4t^2)}{t^2-t+3}dt = 2.844\text{ m}$ → the displacement of the particle

$\int_0^{3.5} |v(t)|dt = \int_0^{3.5} \left|\frac{10\sin(0.4t^2)}{t^2-t+3}\right|dt = 3.737$

→ the distance traveled by the particle

(d) A second particle moves along the x-axis with position given by $x_2(t) = t^2 - t$ for $0 \le t \le 3.5$. At what time t are the two particles moving with the same velocity?

$t^2 - t = \frac{10\sin(0.4t^2)}{t^2-t+3}$

$t = 0,\ 2.000$

Question 2

Overview

In this problem a particle moves along the x-axis. For $0 \le t \le 3.5,$ the velocity of the particle is given by $v(t) = \dfrac{10\sin\left(0.4t^2\right)}{t^2 - t + 3},$ and the particle's position is $x = -5$ at time $t = 0.$ In part (a) students were asked for the acceleration of the particle at time $t = 3.$ A correct response should demonstrate that acceleration is the derivative of velocity and show the evaluation of $v'(3)$ from a graphing calculator. In part (b) students were asked for the position of the particle at time $t = 3.$ A correct response should find the net change in the particle's position as the integral of $v(t)$ across the interval $[0, 3]$ and add this change in position to the particle's position at time $t = 0.$ In part (c) students were asked to evaluate the integrals $\int_0^{3.5} v(t)\,dt$ and $\int_0^{3.5} |v(t)|\,dt$ and to interpret the meaning of each integral in the context of the problem. A correct response should show the values of the two integrals obtained from a graphing calculator and convey that a definite integral of velocity gives the particle's displacement, while a definite integral of speed (i.e., $|v(t)|$) gives the particle's total distance traveled, across the time interval of integration. In part (d) students were given that a second particle moves along the x-axis with position given by $x_2(t) = t^2 - t$ for $0 \le t \le 3.5$ and are asked for the time t when the two particles are moving with the same velocity. A correct response should demonstrate that the second particle's velocity is obtained by differentiating its position function and proceed by solving for when the first particle's velocity, the given $v(t),$ matches $x_2{}'(t)$ within the interval $0 \le t \le 3.5.$

For part (a) see LO 2.3C/EK 2.3C1. For parts (b) and (c), see LO 3.3B(b)/EK 3.3B2, LO 3.4C/EK 3.4C1. For part (d) see LO 2.3C/EK 2.3C1. This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, building notational fluency, and communicating.

Sample: 2A
Score: 9

The response earned all 9 points: 1 point in part (a), 3 points in part (b), 3 points in part (c), and 2 points in part (d). In part (a) the response earned the point with the equation because it gives the correct answer along with the identification of this value as $v'(3).$ In part (b) the response earned the first point with the definite integral on the left side of the equation. The response earned the second point with the addition of -5 to that integral. The response earned the third point with the answer on the right side of the equation. In part (c) the response earned the first point with the values of the two definite integrals. The response earned both of the interpretation points as the two integrals are interpreted as "displacement" and "total distance," and both reference the time interval. In part (d) the response earned the first point in line 1 because the right side of the equation is an expression equivalent to $x_2{}'(t).$ The answer $t = 1.5705$ earned the second point.

Sample: 2B
Score: 6

The response earned 6 points: 1 point in part (a), 3 points in part (b), 2 points in part (c), and no points in part (d). In part (a) the response earned the point with the equation in line 2 because it identifies the correct answer as $v'(3).$ In part (b) the response earned the first point with the definite integral on the left side of the equation. The response earned the second point with the addition of -5 to that integral. The response earned the third point with

Question 2 (continued)

the answer on the right side of the equation. In part (c) the response earned the first point with the values of the two definite integrals. The interpretation of the first integral as "the position of the particle at time $t = 3.5$" is incorrect. The interpretation of the second integral as "the total distance that the particle has traveled from $t = 0$ to $t = 3.5$" is correct. Thus, the response earned 1 of the 2 interpretation points in part (c). In part (d) the response did not earn the first point because $x_2{}'(t)$ is not set equal to $v(t)$. The response also incorrectly indicates "there is not a time where the two particles are moving with the same velocity," so the answer point was not earned.

Sample: 2C
Score: 3

The response earned 3 points: no point in part (a), 1 point in part (b), 2 points in part (c), and no points in part (d). In part (a) the response has an attempt to use the quotient rule to find $v'(3)$. However, the two quantities in the numerator of the stated derivative are reversed and result in the negative of $v'(t)$. This error produces an incorrect value of 2.118, so the point was not earned. In part (b) the response earned the first point with the integral $\int_0^3 v(t)\,dt$. Because there is no use of the initial condition, the answer is incorrect, and no other points in part (b) were earned. In part (c) the response earned the first point with the values of the two definite integrals. The identifications of the first integral as displacement and the second integral as distance traveled are correct, but the response does not reference the time interval in either case. The response earned 1 of the 2 interpretation points in part (c). In part (d) the response has an incorrect equation that leads to an incorrect answer, so no points were earned.

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Question 3

(a) $f(-5) = f(1) + \int_1^{-5} g(x)\,dx = f(1) - \int_{-5}^{1} g(x)\,dx$ $= 3 - \left(-9 - \frac{3}{2} + 1\right) = 3 - \left(-\frac{19}{2}\right) = \frac{25}{2}$	$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$
(b) $\int_1^6 g(x)\,dx = \int_1^3 g(x)\,dx + \int_3^6 g(x)\,dx$ $= \int_1^3 2\,dx + \int_3^6 2(x-4)^2\,dx$ $= 4 + \left[\frac{2}{3}(x-4)^3\right]_{x=3}^{x=6} = 4 + \frac{16}{3} - \left(-\frac{2}{3}\right) = 10$	$3 : \begin{cases} 1 : \text{split at } x = 3 \\ 1 : \text{antiderivative of } 2(x-4)^2 \\ 1 : \text{answer} \end{cases}$
(c) The graph of f is increasing and concave up on $0 < x < 1$ and $4 < x < 6$ because $f'(x) = g(x) > 0$ and $f'(x) = g(x)$ is increasing on those intervals.	$2 : \begin{cases} 1 : \text{intervals} \\ 1 : \text{reason} \end{cases}$
(d) The graph of f has a point of inflection at $x = 4$ because $f'(x) = g(x)$ changes from decreasing to increasing at $x = 4$.	$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$

NO CALCULATOR ALLOWED

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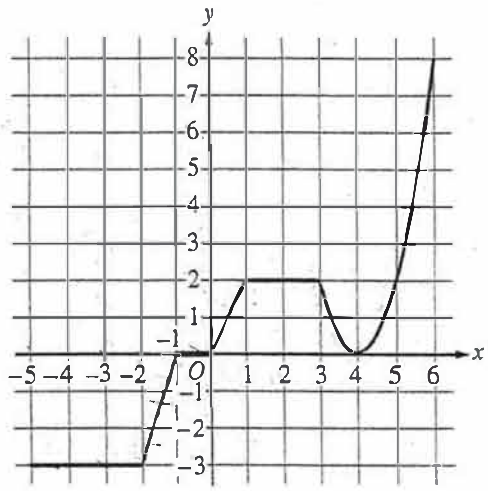

Graph of g

3. The graph of the continuous function g, the derivative of the function f, is shown above. The function g is piecewise linear for $-5 \le x < 3$, and $g(x) = 2(x-4)^2$ for $3 \le x \le 6$.

(a) If $f(1) = 3$, what is the value of $f(-5)$?

$f'(x) = g(x)$

$f(x) = \int_1^x g(t)\,dt + 3$

$f(-5) = -\int_{-5}^{1} g(t)\,dt + 3$

$f(-5) = -\left(\frac{1}{2}(1)(2) - \frac{1}{2}(1)(3) - (3)(3)\right) + 3$

$f(-5) = -(1 - \frac{3}{2} - 9) + 3$

$f(-5) = 8 + \frac{3}{2} + 3$

$f(-5) = 11 + \frac{3}{2}$

(b) Evaluate $\int_1^6 g(x)\,dx$.

$18/3 = 6$

$\int_1^6 g(x)dx = \int_1^3 g(x)dx + \int_3^6 g(x)dx$

$\int_1^6 g(x)dx = (2)(2) + \int_3^6 2(x-4)^2 dx$

$\int_1^6 g(x)dx = 4 + \left(\frac{2}{3}(x-4)^3\right)\Big|_3^6$

$\int_1^6 g(x)dx = 4 + \left(\frac{2}{3}(8) - \frac{2}{3}(-1)\right)$

$\int_1^6 g(x)dx = 4 + \frac{16}{3} + \frac{2}{3}$

$\int_1^6 g(x)dx = 10$

Continue question 3 on page 15.

NO CALCULATOR ALLOWED

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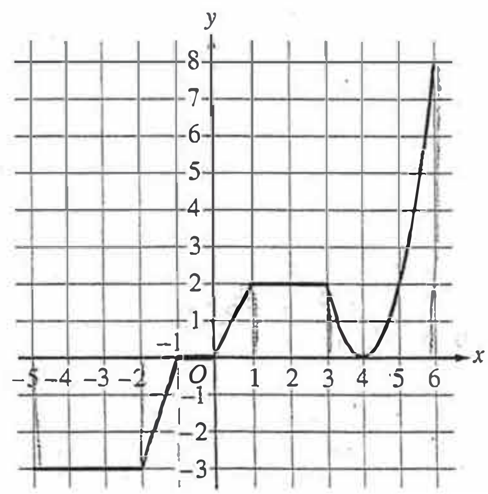
(c) For $-5 < x < 6$, on what open intervals, if any, is the graph of f both increasing and concave up? Give a reason for your answer.

On the interval $(0,1) \cup (4,6)$ f is both increasing and concave up since $f'(x) = g(x)$ and g is positive on the that interval meaning f is increasing on that interval, and g is increasing ~~on~~ that interval, meaning $f''(x) > 0$ on that interval, therefore f is concave up on that interval

(d) Find the x-coordinate of each point of inflection of the graph of f. Give a reason for your answer. A

f has a point of inflection at $x=4$ since $f'(x) = g(x)$ and since g switches from decreasing to increasing at $x=4$, therefore $f''(4) = 0$ at that point and would change signs from ⊖ to ⊕ at $x=4$, therefore $x=4$ is an inflection point.

NO CALCULATOR ALLOWED



Graph of g

$g(x) = f'(x) \quad f(x) = \int g(x)\,dx$

3. The graph of the continuous function g, the derivative of the function f, is shown above. The function g is piecewise linear for $-5 \le x < 3$, and $g(x) = 2(x-4)^2$ for $3 \le x \le 6$.

(a) If $f(1) = 3$, what is the value of $f(-5)$?

$\frac{6}{2} + \frac{19}{2} \quad -\frac{19}{2} - \frac{3}{2}$

$$\int_{-5}^{1} g(x)\,dx = f(1) - f(-5)$$
$$f(-5) = 3 - \int_{-5}^{1} g(x)\,dx$$
$$= 3 - \left[-3(3) - \tfrac{1}{2}(1)(3) + \tfrac{1}{2}(1)(2)\right]$$
$$= 3 - \left[\left(-9 - \tfrac{3}{2}\right) + 1\right]$$
$$= 3 - \left[\tfrac{-21}{2} + \tfrac{2}{2}\right] = 3 + \tfrac{19}{2} = \tfrac{25}{2}$$

(b) Evaluate $\int_1^6 g(x)\,dx$.

$u = x - 4$
$du = dx$

$$\int_1^6 g(x)\,dx = 3(3) + \int_3^6 g(x)\,dx$$
$$= 9 + \int_3^6 2(x-4)^2\,dx \qquad \frac{27}{3}$$
$$= 9 + 2\int_{-1}^{2} u^2\,du$$
$$= 9 + 2 \cdot \tfrac{1}{3}u \Big|_{-1}^{2}$$
$$= 9 + 2\left(\tfrac{2}{3} - \tfrac{1}{3}\right) = 9 + \tfrac{2}{3} = \tfrac{29}{3}$$

Continue question 3 on page 15.

NO CALCULATOR ALLOWED

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(c) For $-5 < x < 6$, on what open intervals, if any, is the graph of f both increasing and concave up? Give a reason for your answer.

f' is pos f'' pos

f is Increasing when $f'(x) = g(x)$ is positive
f is Concave up when $f'(x) = g(x)$ is increasing

$\Rightarrow$ The graph of f is concave up and increasing on $(0,1) \cup (4,6)$.

(d) Find the x-coordinate of each point of inflection of the graph of f. Give a reason for your answer. a

f has a point of inflection when $f'(x) = g(x)$ has a maximum or minimum (local)

$\Rightarrow x = 4$

NO CALCULATOR ALLOWED

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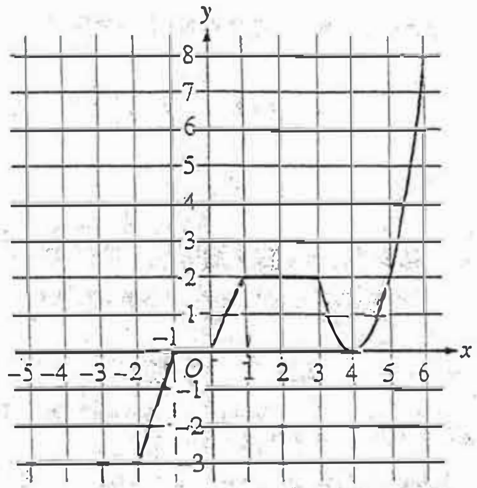
1 of 2



Graph of g $f'(x)$ Œ

3. The graph of the continuous function g, the derivative of the function f, is shown above. The function g is piecewise linear for $-5 \le x < 3$, and $g(x) = 2(x-4)^2$ for $3 \le x \le 6$.

(a) If $f(1) = 3$, what is the value of $f(-5)$? $f'(x) = g(x)$

$f'(x) = 2(x-4)^2$

$f(x) = \frac{2(x-4)^3}{3}$

$f(-5) = \frac{2(-5-4)^3}{3}$

(b) Evaluate $\int_1^6 g(x)\,dx$.

$\int_1^6 g(x)\,dx$

$\left.\frac{2(x-4)^3}{3}\right]_1^6$ $\frac{16}{3} + \frac{54}{3} = \boxed{\frac{70}{3}}$

Continue question 3 on page 15.

(c) bFor $-5 < x < 6$, on what open intervals, if any, is the graph of f both increasing and concave up? Give a reason for your answer.

from $0 < x < 1$ and $4 < x < 6$
the graph of f is both inc.
and concave up because
f' is above the x-axis (positive)
and has an increasing slope

(d) Find the x-coordinate of each point of inflection of the graph of f. Give a reason for your answer.

there is a point of inflection at
$x = 4$ because the slope of f'
changes from decreasing to increasing
(−) (+)

Question 3

Overview

In this problem the graph of the continuous function g is provided; g is piecewise linear for $-5 \le x < 3,$ and $g(x) = 2(x - 4)^2$ for $3 \le x \le 6.$ It is also given that g is the derivative of the function *f*. In part (a) students were given that $f(1) = 3$ and asked for the value of $f(-5).$ A correct response should demonstrate knowledge that *f* is an antiderivative of *g*, so that $f(-5) = f(1) + \int_1^{-5} g(x)\,dx.$ The integral $\int_1^{-5} g(x)\,dx$ should then be evaluated using properties of definite integrals and computation of areas of the regions between the graph of g and the *x*-axis using geometry. In part (b) students were asked to evaluate $\int_1^6 g(x)\,dx.$ A correct response should use the property of integrals to split the interval of integration into the sum of integrals across adjacent intervals $[1, 3]$ and $[3, 6].$ One of the resulting integrals can be computed using geometry and the other using an antiderivative of $g(x) = 2(x - 4)^2$ on the interval $3 \le x \le 6.$ In part (c) students were asked for the open intervals on $-5 < x < 6$ where the graph of *f* is both increasing and concave up and to give a reason for their answer. A correct response should demonstrate the connection between properties of the derivative of *f* and the properties of monotonicity and concavity for the graph of *f*. The graph of *f* is strictly increasing where $g = f'$ is positive, and the graph of g is concave up where the graph of $g = f'$ is increasing. In part (d) students were asked for the *x*-coordinate of each point of inflection of the graph of *f* and to give a reason for their answer. A correct response should convey that a point of inflection of the graph of *f* occurs at a point where the derivative of *f* changes from increasing to decreasing, or from decreasing to increasing. This can be obtained from the supplied graph of $g = f',$ which changes from decreasing to increasing at $x = 4.$

For part (a) see LO 3.2C/EK 3.2C1, LO 3.2C/EK 3.2C2. For part (b) see LO 3.2C/EK 3.2C1, LO 3.2C/EK 3.2C2, LO 3.3B(b)/EK 3.3B2, LO 3.3B(b)/EK 3.3B5. For parts (c) and (d), see LO 2.2A/EK 2.2A1. This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, connecting multiple representations, building notational fluency, and communicating.

Sample: 3A
Score: 9

The response earned all 9 points: 2 points in part (a), 3 points in part (b), 2 points in part (c), and 2 points in part (d). In part (a) the response earned the first point with the expression $-\int_{-5}^{1} g(t)\,dt$ in line 3 on the left. The second point would have been earned by the numerical expression in line 4 with no simplification. In this case, correct simplification to $11 + \frac{3}{2}$ earned the second point. In part (b) the response earned the first point with the sum of the two integrals in line 1 on the left. The second point was earned with the antiderivative expression $\left(\frac{2}{3}(x - 4)^3\right)$ in line 3. The second point would have been earned by the numerical expression in line 4 with no simplification. In this case, correct simplification to 10 earned the third point. In part (c) the union of intervals "$(0, 1) \cup (4, 6)$" earned the first point. The second point was earned with the reason $f'(x) = g(x),$ "g is positive," and "g is increasing on that interval." In part (d) the first point was earned by identifying the *x*-coordinate of a point of inflection at $x = 4.$ The second point was earned with the reason $f'(x) = g(x)$ and "g switches from decreasing to increasing at $x = 4.$"

Question 3 (continued)

Sample: 3B
Score: 6

The response earned 6 points: 2 points in part (a), no points in part (b), 2 points in part (c), and 2 points in part (d). In part (a) the response earned the first point with the expression $\int_{-5}^{1} g(x)\,dx$ in line 1. The second point would have been earned by the numerical expression in line 3 with no simplification. In this case, correct simplification to $\frac{25}{2}$ earned the second point. In part (b) the integral expression $3(3) + \int_{3}^{6} g(x)\,dx$ in line 1 did not earn the first point because $\int_{1}^{3} g(x)\,dx = 4,$ not 9. This response used substitution of variables to write $\int_{3}^{6} 2(x-4)^2\,dx$ in an equivalent form. The antiderivative in line 4 is incorrect, and the response did not earn the second point. As a result of this error, the response is not eligible for the answer point. In part (c) the union of intervals "$(0, 1) \cup (4, 6)$" earned the first point. The second point was earned with the reason "$f'(x) = g(x)$ is positive" in line 1 and "$f'(x) = g(x)$ is increasing" in line 2. In part (d) the first point was earned by identifying the x-coordinate of a point of inflection at $x = 4.$ The second point was earned with the reason "$f'(x) = g(x)$ has a maximum or minimum (local)."

Sample: 3C
Score: 3

The response earned 3 points: no points in part (a), 1 point in part (b), 1 point in part (c), and 1 point in part (d). In part (a) an integral expression is not presented nor is its numerical equivalent, so the first point was not earned. The value given for $f(-5)$ is incorrect, so the second point was not earned. In part (b) the response did not earn the first point because $\int_{1}^{6} g(x)\,dx$ is not written as the sum of two integrals or the equivalent. The second point was earned with the antiderivative expression in line 2. Because the second point was earned, the response is eligible for the third point. The answer is incorrect, however, so the third point was not earned. In part (c) the first point was earned with the intervals "$0 < x < 1$ and $4 < x < 6.$" Although "f' is above the x-axis" is a valid reason for why f is increasing on those intervals, f' "has an increasing slope" is not a valid reason to explain why the graph of f is concave up on those intervals. The second point was not earned. In part (d) the first point was earned by identifying the x-coordinate of a point of inflection at $x = 4.$ "The slope of f' changes from decreasing to increasing" is not a valid reason to explain why the graph of f has a point of inflection at $x = 4,$ so the second point was not earned.

2018

AP CollegeBoard

AP Calculus AB

Sample Student Responses and Scoring Commentary

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☑ Scoring Guideline

☑ Student Samples

☑ Scoring Commentary

Question 4

(a) $H'(6) \approx \dfrac{H(7)-H(5)}{7-5} = \dfrac{11-6}{2} = \dfrac{5}{2}$ $H'(6)$ is the rate at which the height of the tree is changing, in meters per year, at time $t = 6$ years.	$2: \begin{cases} 1: \text{estimate} \\ 1: \text{interpretation with units} \end{cases}$
(b) $\dfrac{H(5)-H(3)}{5-3} = \dfrac{6-2}{2} = 2$ Because H is differentiable on $3 \le t \le 5$, H is continuous on $3 \le t \le 5$. By the Mean Value Theorem, there exists a value c, $3 < c < 5$, such that $H'(c) = 2$.	$2: \begin{cases} 1: \dfrac{H(5)-H(3)}{5-3} \\ 1: \text{conclusion using} \\ \quad\ \text{Mean Value Theorem} \end{cases}$
(c) The average height of the tree over the time interval $2 \le t \le 10$ is given by $\dfrac{1}{10-2}\displaystyle\int_2^{10} H(t)\,dt$. $\dfrac{1}{8}\displaystyle\int_2^{10} H(t)\,dt \approx \dfrac{1}{8}\left(\dfrac{1.5+2}{2}\cdot 1 + \dfrac{2+6}{2}\cdot 2 + \dfrac{6+11}{2}\cdot 2 + \dfrac{11+15}{2}\cdot 3\right)$ $= \dfrac{1}{8}(65.75) = \dfrac{263}{32}$ The average height of the tree over the time interval $2 \le t \le 10$ is $\dfrac{263}{32}$ meters.	$2: \begin{cases} 1: \text{trapezoidal sum} \\ 1: \text{approximation} \end{cases}$
(d) $G(x) = 50 \Rightarrow x = 1$ $\dfrac{d}{dt}(G(x)) = \dfrac{d}{dx}(G(x)) \cdot \dfrac{dx}{dt} = \dfrac{(1+x)100 - 100x \cdot 1}{(1+x)^2} \cdot \dfrac{dx}{dt} = \dfrac{100}{(1+x)^2} \cdot \dfrac{dx}{dt}$ $\left.\dfrac{d}{dt}(G(x))\right\rvert_{x=1} = \dfrac{100}{(1+1)^2} \cdot 0.03 = \dfrac{3}{4}$ According to the model, the rate of change of the height of the tree with respect to time when the tree is 50 meters tall is $\dfrac{3}{4}$ meter per year.	$3: \begin{cases} 2: \dfrac{d}{dt}(G(x)) \\ 1: \text{answer} \end{cases}$ Note: max $1/3$ [1-0] if no chain rule

4A
1 of 2

t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15

4. The height of a tree at time t is given by a twice-differentiable function H, where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table above.

(a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.

$$H'(6) \approx \frac{\Delta H(t)}{\Delta t} \approx \frac{H(7)-H(5)\,m}{(7-5)_{yr}} = \frac{(11-6)m}{(7-5)_y} = \boxed{\frac{5 \text{ meters}}{2 \text{ years}}}$$

When $t = 6$ years, the rate at which the tree is growing is $H'(6)$ meters per year

(b) Explain why there must be at least one time t, for $2 < t < 10$, such that $H'(t) = 2$.

By the MVT, as $H(t)$ is continuous and differentiable on $t \in (2,10)$, there must be $H'(c) = 2$ where $2 < c < 10$ if there exists $\frac{H(b)-H(a)}{b-a} = 2$ on the interval $(2,10)$. $\frac{H(5)-H(3)}{(5-3)\text{years}} = \frac{6m-2m}{2\text{ years}} = 2\,m/yr$

So c exists on interval $c \in (2,10)$

Continue question 4 on page 17.

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(c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the average height of the tree over the time interval $2 \le t \le 10$.

Total height: $\frac{1}{2}\left(1\,(1.5+2) + 2\,(2+6) + 2\,(6+11) + 3\,(11+15)\right)$

Average height: $\frac{1}{10_y - 2_{yr}}$ total ht $= \boxed{\frac{1}{8} \times \frac{1}{2}\left(3.5 + 2(8) + 2(17) + 3(26)\right) \text{ meters}}$

(d) The height of the tree, in meters, can also be modeled by the function G, given by $G(x) = \frac{100x}{1+x}$, where x is the diameter of the base of the tree, in meters. When the tree is 50 meters tall, the diameter of the base of the tree is increasing at a rate of 0.03 meter per year. According to this model, what is the rate of change of the height of the tree with respect to time, in meters per year, at the time when the tree is 50 meters tall?

$G(x) = \frac{100x}{1+x}$ $\qquad G(x) = 50 \qquad \frac{dx}{dt} = 0.03$ m/y

$x = 1$ m

$G'(x) = \frac{100\frac{dx}{dt}(1+x) - (\frac{dx}{dt})100x}{(1+x)^2}$

$\frac{100 \times 0.03\,(2) - 0.03 \times 100}{4}$

$50 = \frac{100x}{1+x} \Rightarrow \cancel{50}(1+x) = \cancel{100}^{2}x$

$\Rightarrow 1 = x \quad x = 1$ m

$\frac{6-3}{4} = \boxed{\frac{3}{4} \text{ m/year}}$

t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15

4. The height of a tree at time t is given by a twice-differentiable function H, where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table above.

(a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.

$$H'(6) = \frac{H(7) - H(5)}{7-5} = \frac{11-6}{2} = \boxed{\frac{5}{2} \text{ meters/year}}$$

H'(6) is the rate that the tree is growing, in meters per year, at t = 6 years.

(b) Explain why there must be at least one time t, for $2 < t < 10$, such that $H'(t) = 2$.

H is twice - differentiable, which means it is also continuous. Therefore, the MVT guarantees that H'(t) = 2 since

$$\frac{H(10) - H(2)}{10-2} = 2$$

Continue question 4 on page 17.

NO CALCULATOR ALLOWED

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(c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the average height of the tree over the time interval $2 \le t \le 10$.

$(1)\left(\frac{1.5+2}{2}\right)+(2)\left(\frac{2+6}{2}\right)+(2)\left(\frac{6+11}{2}\right)+(3)\left(\frac{11+15}{2}\right)$

$\frac{3.5}{2}+\frac{16}{2}+\frac{34}{2}+\frac{78}{2}$

$= \boxed{\frac{131.5}{2} \text{ meters}}$

(d) The height of the tree, in meters, can also be modeled by the function G, given by $G(x) = \frac{100x}{1+x}$, where x is the diameter of the base of the tree, in meters. When the tree is 50 meters tall, the diameter of the base of the tree is increasing at a rate of 0.03 meter per year. According to this model, what is the rate of change of the height of the tree with respect to time, in meters per year, at the time when the tree is 50 meters tall?

$50 = \frac{100x}{1+x}$

$50(1+x) = 100x$

$50 + 50x = 100x$

$50 = 50x$

$x = 1$

$\frac{dG}{dt} = \frac{(1+x)(100\frac{dx}{dt}) - (100x)(\frac{dx}{dt})}{(1+x)^2}$

$\frac{(1+1)(100(0.03)) - (100(1))(0.03)}{(1+1)^2}$

$\frac{(2)(3)-3}{4} = \frac{6-3}{4} = \boxed{\frac{3}{4} \text{ meters/year}}$

NO CALCULATOR ALLOWED

4C
1 of 2

t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15

4. The height of a tree at time t is given by a twice-differentiable function H, where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table above.

(a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.

$$H'(6) = \frac{H(7) - H(5)}{7-5} = \frac{11-6}{7-5} \quad \frac{5}{2}$$

H'(6) is the rate in m/year in which the height of a tree increases.

(b) Explain why there must be at least one time t, for $2 < t < 10$, such that $H'(t) = 2$.

Then must be one time t for 2 < t < 10 that H'(t) = 2 b/c H(t) is continuous and differentiable

Continue question 4 on page 17.

NO CALCULATOR ALLOWED

4C
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(c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the average height of the tree over the time interval $2 \le t \le 10$.

$\frac{1}{2}(1.5+2) + \frac{1}{2}(2+6)2 + \frac{1}{2}(6+11)2 + \frac{1}{2}(11+15)3$

$= \frac{3.5}{2} + 8 + 17 + 39$

$= \frac{7}{2} + \frac{16}{2} + \frac{34}{2} + \frac{78}{2}$

$= \frac{135}{2}$

(d) The height of the tree, in meters, can also be modeled by the function G, given by $G(x) = \frac{100x}{1+x}$, where x is the diameter of the base of the tree, in meters. When the tree is 50 meters tall, the diameter of the base of the tree is increasing at a rate of 0.03 meter per year. According to this model, what is the rate of change of the height of the tree with respect to time, in meters per year, at the time when the tree is 50 meters tall?

$r = \frac{d}{2}$

$G'(x) = \frac{(1+x)(100) - 100x}{(1+x)^2}$

$A = bh + \pi r^2$

$A = dh + \pi\left(\frac{d}{2}\right)^2$

$A = x\left(\frac{100x}{1+x}\right) + \pi\left(\frac{x}{2}\right)^2$

GO ON TO THE NEXT PAGE.

Question 4

Overview

The context of this problem is a tree, the height of which at time t is given by a twice-differentiable function H, where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are provided in a table. In part (a) students were asked to use the tabular data to estimate $H'(6)$ and then to interpret the meaning of $H'(6)$, using correct units, in the context of the problem. The correct response should estimate the derivative value using a difference quotient, drawing from data in the table that most tightly bounds $t = 6$. In part (b) students were asked to explain why there must be at least one time t, for $2 < t < 10$, such that $H'(t) = 2$. A correct response should demonstrate that the Mean Value Theorem applies to H on the interval $[3, 5]$, over which the average rate of change of H (using data from the table) is $\frac{6-2}{2} = 2$. In part (c) students were asked to use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the average height of the tree over the time interval $2 \le t \le 10$. A correct response should demonstrate that the average height of the tree for $2 \le t \le 10$ is given by dividing the definite integral of H across the interval by the width of the interval. The value of the integral $\int_2^{10} H(t)\, dt$ is to be approximated using a trapezoidal sum and data in the table. In part (d) students were given another model for the tree's height, in meters, $G(x) = \frac{100x}{1+x}$, where x is the diameter of the base of the tree, in meters. It is further given that when the tree is 50 meters tall, it is growing so that the diameter at the base of the tree is increasing at the rate of 0.03 meter per year. Using this model, students were asked to find the rate of change of the height of the tree with respect to time, in meters per year, at the time when the tree is 50 meters tall. A correct response should apply the chain rule to obtain that $\frac{dG}{dt} = \frac{dG}{dx} \cdot \frac{dx}{dt}$. The derivative expression $\frac{dG}{dx}$ can be obtained from the given expression for $G(x)$ using derivative rules (e.g., the quotient rule) and the value of $\frac{dx}{dt}$ at the instant in question provided in the problem statement.

For part (a) see LO 2.1B/EK 2.1B1, LO 2.3A/EK 2.3A1, LO 2.3A/EK 2.3A2. For part (b) see LO 2.4A/EK 2.4A1. For part (c) see LO 3.2B/EK 3.2B2, LO 3.4B/EK 3.4B1. For part (d) see LO 2.1C/EK 2.1C3, LO 2.1C/EK 2.1C4, LO 2.3C/EK 2.3C2. This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, connecting multiple representations, building notational fluency, and communicating.

Sample: 4A
Score: 9

The response earned all 9 points: 2 points in part (a), 2 points in part (b), 2 points in part (c), and 3 points in part (d). In part (a) the response would have earned the first point for $\frac{(11-6)}{(7-5)}$ with no simplification. In this case, correct simplification to $\frac{5}{2}$ earned the point. Although not required for the first point, the answer includes the correct units of "meters/years" which is considered for the second point. The response earned the second point for the interpretation that includes the three necessary elements: an interpretation of H' as a rate in the context of the tree, correct units of meters per year, and an interpretation of the input "6" as the moment in time of $t = 6$ years. In part (b) the response earned the first point for the difference quotient $\frac{H(5) - H(3)}{(5-3)}$ that appears in line 4. The first point does not require the substitution of function values and simplification that follows; this work is

Question 4 (continued)

considered in the context of the second point. The response earned the second point for the explanation using the Mean Value Theorem (MVT) and the difference quotient on the interval $[3, 5]$. The response explicitly states in line 1 that $H(t)$ is continuous which is necessary to earn the point. The units displayed are correct but not required to earn either the first or second points. In part (c) the response earned the first point for the trapezoidal sum labeled "Total height" with no simplification. The response earned the second point with the boxed answer with no simplification. Note that both points were earned in part (c) without simplification of numerical answers. The units given with the average height are correct but not required to earn the second point. In part (d) the response earned the first 2 points with the correct derivative in line 2: $\dfrac{100\frac{dx}{dt}(1+x) - \left(\frac{dx}{dt}\right)100x}{(1+x)^2}$. The use of $G'(x)$ rather than $\dfrac{dG}{dt}$ notation does not impact the points earned. The response would have earned the third point for $\dfrac{100 \times 0.03(2) - 0.03 \times 100}{4}$ with no simplification. In this case, correct simplification to $\dfrac{3}{4}$ earned the third point. The response includes correct units that are not required to earn the third point.

Sample: 4B
Score: 6

The response earned 6 points: 2 points in part (a), no points in part (b), 1 point in part (c), and 3 points in part (d). In part (a) the response would have earned the first point for $\dfrac{11-6}{2}$ with no simplification. In this case, correct simplification to $\dfrac{5}{2}$ earned the point. Although not required for the first point, the answer includes the correct units of meters/year which is considered for the second point. The response earned the second point for the interpretation that includes the three necessary elements: an interpretation of H' as a rate in the context of the tree, correct units of meters per year, and an interpretation of the input "6" as the moment in time of $t = 6$ years. In part (b) the response did not earn the first point because the difference quotient does not use the interval $[3, 5]$ that results in a secant slope of 2. The response did not earn the second point because, although the Mean Value Theorem (MVT) is cited along with the continuity of H, there is no explanation connecting the Mean Value Theorem to the values of $H(t)$ in the table. In part (c) the response earned the first point for the trapezoidal sum in line 1 with no simplification. The arithmetic and simplification that follow are considered for the second point. The second point was not earned because the sum is not multiplied by $\dfrac{1}{8}$ to find the average height of the tree on the interval $[2, 10]$. In part (d) the response earned the first 2 points with the correct derivative in line 1 on the left: $\dfrac{dG}{dt} = \dfrac{(1+x)\left(100\frac{dx}{dt}\right) - (100x)\left(\frac{dx}{dt}\right)}{(1+x)^2}$. The response would have earned the third point for $\dfrac{(1+1)(100(0.03)) - (100(1))(0.03)}{(1+1)^2}$ in line 2 on the left with no simplification. In this case, correct simplification to $\dfrac{3}{4}$ earned the third point. The response includes correct units that are not required to earn the third point.

Question 4 (continued)

Sample: 4C
Score: 3

The response earned 3 points: 1 point in part (a), no points in part (b), 1 point in part (c), and 1 point in part (d). In part (a) the response would have earned the first point for $\frac{11-6}{7-5}$ with no simplification. In this case, correct simplification to $\frac{5}{2}$ earned the point. The response did not earn the second point because the interpretation of $H'(6)$ does not include an interpretation of the input "6" as the moment in time of $t = 6$ years. In part (b) the response did not earn the first point because no difference quotient is given. The response is not eligible for the second point because the explanation given does not reference the interval $[3, 5]$ that results in the secant slope of 2. Although the hypotheses of the Mean Value Theorem are stated, the conclusion is not. In part (c) the response earned the first point for the trapezoidal sum in line 1 with no simplification. The arithmetic and simplification that follow are considered for the second point. The response did not earn the second point. There is an arithmetic error in line 3, and the sum is not multiplied by $\frac{1}{8}$ to find the average height of the tree on the interval $[2, 10]$. In part (d) the response earned 1 of the first 2 points for the correct derivative of G with respect to x in line 2 on the right. Because the derivative does not include the chain rule, the response is not eligible for additional points in part (d).

2018

AP CollegeBoard

AP Calculus AB

Sample Student Responses and Scoring Commentary

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Question 5

(a) The average rate of change of f on the interval $0 \le x \le \pi$ is $\dfrac{f(\pi) - f(0)}{\pi - 0} = \dfrac{-e^{\pi} - 1}{\pi}$.

1 : answer

(b) $f'(x) = e^x \cos x - e^x \sin x$

$f'\left(\frac{3\pi}{2}\right) = e^{3\pi/2} \cos\left(\frac{3\pi}{2}\right) - e^{3\pi/2} \sin\left(\frac{3\pi}{2}\right) = e^{3\pi/2}$

The slope of the line tangent to the graph of f at $x = \frac{3\pi}{2}$ is $e^{3\pi/2}$.

$2 : \begin{cases} 1 : f'(x) \\ 1 : \text{slope} \end{cases}$

(c) $f'(x) = 0 \Rightarrow \cos x - \sin x = 0 \Rightarrow x = \frac{\pi}{4}, x = \frac{5\pi}{4}$

x	$f(x)$
0	1
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}} e^{\pi/4}$
$\frac{5\pi}{4}$	$-\frac{1}{\sqrt{2}} e^{5\pi/4}$
2π	$e^{2\pi}$

The absolute minimum value of f on $0 \le x \le 2\pi$ is $-\frac{1}{\sqrt{2}} e^{5\pi/4}$.

$3 : \begin{cases} 1 : \text{sets } f'(x) = 0 \\ 1 : \text{identifies } x = \frac{\pi}{4}, x = \frac{5\pi}{4} \\ \quad \text{as candidates} \\ 1 : \text{answer with justification} \end{cases}$

(d) $\lim\limits_{x \to \pi/2} f(x) = 0$

Because g is differentiable, g is continuous.

$\lim\limits_{x \to \pi/2} g(x) = g\left(\frac{\pi}{2}\right) = 0$

By L'Hospital's Rule,

$\lim\limits_{x \to \pi/2} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to \pi/2} \dfrac{f'(x)}{g'(x)} = \dfrac{-e^{\pi/2}}{2}$.

$3 : \begin{cases} 1 : g \text{ is continuous at } x = \frac{\pi}{2} \\ \quad \text{and limits equal } 0 \\ 1 : \text{applies L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$

Note: max $1/3$ [1-0-0] if no limit notation attached to a ratio of derivatives

NO CALCULATOR ALLOWED

5A
1 of 2

5. Let f be the function defined by $f(x) = e^x \cos x$.

(a) Find the average rate of change of f on the interval $0 \le x \le \pi$.

$$\frac{\int_0^{\pi} f'(x)\,dx}{\pi} = \frac{f(\pi) - f(0)}{\pi} = \frac{e^{\pi}\cos \pi - e^0 \cos 0}{\pi}$$

$$= \frac{e^{\pi}(-1) - 1(1)}{\pi} = \boxed{\frac{-e^{\pi} - 1}{\pi}}$$

(b) What is the slope of the line tangent to the graph of f at $x = \frac{3\pi}{2}$?

$$\frac{df}{dx} = e^x \cos x + e^x(-\sin x)$$

$$= e^x(\cos x - \sin x)$$

$$e^{3\pi/2}(0 - (-1)) = \boxed{e^{3\pi/2}}$$

Continue question 5 on page 19.

NO CALCULATOR ALLOWED

5A
2 of 2

(c) Find the absolute minimum value of f on the interval $0 \le x \le 2\pi$. Justify your answer.

$f'(x) = e^x(\cos x - \sin x) = 0 \rightarrow e^x = 0$ or $\cos x = \sin x$
$x = \frac{\pi}{4}, \frac{5\pi}{4}$.

From part B

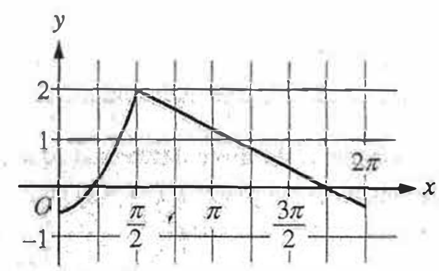
Minima either have a derivative of 0, undefined, or are bordering points.

So, possible values: $x = 0, x = 2\pi, x = \frac{\pi}{4}$ $x = \frac{5\pi}{4}$

$f(x) = 1$ $f(x) = e^{2\pi}$ $f(x) = e^{\pi/4}\left(\frac{\sqrt{2}}{2}\right)$ $f(x) = e^{5\pi/4}\left(-\frac{\sqrt{2}}{2}\right)$.

Since $e^{5\pi/4}\left(-\frac{\sqrt{2}}{2}\right)$ is the only negative value of $f(x)$, the absolute minimum of $f(x)$ on $0 \le x \le 2\pi$ is $\boxed{-\frac{\sqrt{2}}{2}e^{5\pi/4}}$.

(d) Let g be a differentiable function such that $g\left(\frac{\pi}{2}\right) = 0$. The graph of g', the derivative of g, is shown below. Find the value of $\lim_{x \to \pi/2} \frac{f(x)}{g(x)}$ or state that it does not exist. Justify your answer.



Graph of g'

Note $\lim_{x \to \frac{\pi}{2}} (g(x)) = 0$ since $g(x)$ is differentiable (and thus continuous), and $g(\frac{\pi}{2}) = 0$. Also $f(\frac{\pi}{2}) = e^{\pi/2}(\cos(\frac{\pi}{2})) = e^{\pi/2}(0) = 0$, and $f(x)$ is continuous, so $\lim_{x \to \frac{\pi}{2}} f(x) = 0$. So, by L'Hopital's rule,

$$\lim_{x \to \frac{\pi}{2}} \left(\frac{f(x)}{g(x)}\right) = \lim_{x \to \frac{\pi}{2}} \left(\frac{f'(x)}{g'(x)}\right) = \frac{e^{\pi/2}\left(\cos\frac{\pi}{2} - \sin\frac{\pi}{2}\right)}{2} = \boxed{\frac{-e^{\pi/2}}{2}}$$

GO ON TO THE NEXT PAGE.

NO CALCULATOR ALLOWED

5B

1 of 2

5. Let f be the function defined by $f(x) = e^x \cos x$.

(a) Find the average rate of change of f on the interval $0 \le x \le \pi$.

$$\frac{f(\pi) - f(0)}{\pi - 0} = \frac{-e^{\pi} - 1}{\pi} = \boxed{-\frac{e^{\pi}}{\pi} - \frac{1}{\pi}}$$

(b) What is the slope of the line tangent to the graph of f at $x = \frac{3\pi}{2}$?

$$f'(x) = -e^x \sin x + e^x \cos x$$

$$f'\left(\frac{3\pi}{2}\right) = -e^{3\pi/2}(-1) + e^{3\pi/2}(0)$$

$$= \boxed{e^{3\pi/2}}$$

Continue question 5 on page 19.

(c) Find the absolute minimum value of f on the interval $0 \le x \le 2\pi$. Justify your answer.

$f'(x) = -e^x \sin x + e^x \cos x = 0$

$\cancel{-e^x} \sin x = \cancel{-e^x} \cos x$

$x = \pm \frac{\pi}{4}$

$f(0) = 1$

$f(2\pi) = e^{2\pi}$

$f\left(-\frac{\pi}{4}\right) = e^{-\pi/4}\left(-\frac{\sqrt{2}}{2}\right)$

$f\left(\frac{\pi}{4}\right) = e^{\pi/4}\left(\frac{\sqrt{2}}{2}\right)$

~~The absolute~~ minimum value is $-\frac{\sqrt{2}e^{-\pi/4}}{2}$ because $\pm\frac{\pi}{4}$ are critical values and $f(-\pi/4) > f(0), f(2\pi)$, and $f(\pi/4)$

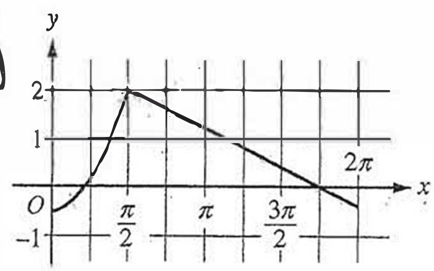
(d) Let g be a differentiable function such that $g\left(\frac{\pi}{2}\right) = 0$. The graph of g', the derivative of g, is shown below. Find the value of $\lim\limits_{x \to \pi/2} \frac{f(x)}{g(x)}$ or state that it does not exist. Justify your answer.

$\lim\limits_{x \to \pi/2} \frac{f'(x)}{g'(x)} = \boxed{-\frac{e^{\pi/2}}{2}}$

$g'(\pi/2) = 2$

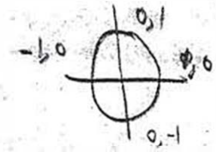
$f'(\pi/2) = -e^{\pi/2}(1) + \cancel{e^{\pi/2}\cos}$

$= -e^{\pi/2}$



Graph of g'

5. Let f be the function defined by $f(x) = e^x \cos x$.



(a) Find the average rate of change of f on the interval $0 \le x \le \pi$.

$$\frac{f(\pi) - f(0)}{\pi - 0} = \frac{e^{\pi}\cos(\pi) - e^{0}\cos(0)}{\pi}$$

$$= \frac{-e^{\pi} - 1}{\pi}$$

(b) What is the slope of the line tangent to the graph of f at $x = \frac{3\pi}{2}$?

$f(\frac{3\pi}{2}) = e^{\frac{3\pi}{2}} \cos(\frac{3\pi}{2})$

$= 0$

$f'(x) = e^x \cos(x) - e^x \sin(x)$

$f'(\frac{3\pi}{2}) = e^{\frac{3\pi}{2}} \cos(\frac{3\pi}{2}) - e^{\frac{3\pi}{2}} \sin(\frac{3\pi}{2})$

$e^{\frac{3\pi}{2}}(0) - e^{\frac{3\pi}{2}}(1)$

$= -e^{\frac{3\pi}{2}}$

$y = -e^{\frac{3\pi}{2}}(x - \frac{3\pi}{2})$

Continue question 5 on page 19.

5C
2 of 2

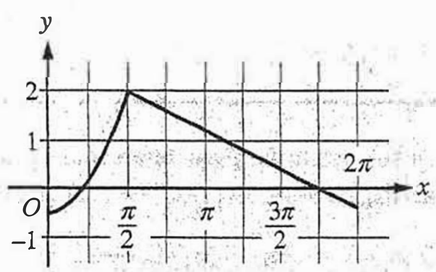
(c) Find the absolute minimum value of f on the interval $0 \le x \le 2\pi$. Justify your answer.

$f'(x) = e^x \cos(x) - e^x \sin(x) = 0$

$e^x \left[\cos(x) - \sin(x)\right) = 0$

the absolute min value is when $f'(x) = 0$ and changes signs from $-$ to $+$

(d) Let g be a differentiable function such that $g\left(\frac{\pi}{2}\right) = 0$. The graph of g', the derivative of g, is shown below. Find the value of $\lim_{x \to \pi/2} \frac{f(x)}{g(x)}$ or state that it does not exist. Justify your answer.



Graph of g'

$\lim_{x \to \frac{\pi}{2}} f(x) = f'\left(\frac{\pi}{2}\right) = -e^{\frac{\pi}{2}}$

$\lim_{x \to \frac{\pi}{2}} g(x) = 2$

$g\left(\frac{\pi}{2}\right) = 0$

limit exists b/c the

$\lim_{x \to \frac{\pi}{2}} \frac{f(x)}{g(x)} = \lim_{x \to \frac{\pi}{2}} \frac{-e^{\frac{\pi}{2}}}{2}$

GO ON TO THE NEXT PAGE.

Question 5

Overview

In this problem the function f is defined by $f(x) = e^x \cos x$. In part (a) students were asked for the average rate of change of f on the interval $0 \le x \le \pi$. A correct response should demonstrate that the average rate of change is found using a difference quotient, $\frac{f(\pi) - f(0)}{\pi - 0}$. In part (b) students were asked for the slope of the line tangent to the graph of f at $x = \frac{3\pi}{2}$. A correct response should use the fact that the slope of the tangent line is the value of the derivative of f at the indicated point. The given expression for $f(x)$ can be differentiated using the product rule. In part (c) students were asked to find, with justification, the absolute minimum value of f on the interval $0 \le x \le 2\pi$. The Extreme Value Theorem guarantees that f attains a minimum on the interval $0 \le x \le 2\pi$. Candidates for locations of this minimum value are the endpoints of the interval and critical points for f inside the interval. In this case, f is differentiable, so critical points are the two zeros of f', $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$. The absolute minimum value of f is the least of the values of f at the four candidates. In part (d) it is given that g is a differentiable function such that $g\left(\frac{\pi}{2}\right) = 0$, and the graph of the derivative of g for $0 \le x \le 2\pi$ is supplied. Students were asked to find the value of $\lim_{x \to \pi/2} \frac{f(x)}{g(x)}$, if it exists, and to justify their answer. A correct response should start with a confirmation that the requested limit is an indeterminate form to which L'Hospital's Rule applies. The numerator, $f(x)$, has limit 0 as $x \to \frac{\pi}{2}$, using limit properties and continuity of the exponential and cosine functions. Because g is differentiable, it is continuous, so the limit of the denominator, $g(x)$, can be computed by substitution, yielding $\lim_{x \to \pi/2} g(x) = 0$. After the indeterminate form is confirmed, applying L'Hospital's Rule leads to the limit $\lim_{x \to \pi/2} \frac{f'(x)}{g'(x)}$, which can be evaluated using properties of limits and the provided graph of g'.

For part (a) see LO 2.1A/EK 2.1A1. For part (b) see LO 2.1C/EK 2.1C3, LO 2.3B/EK 2.3B1. For part (c) see LO 1.2B/EK 1.2B1, LO 2.2A/EK 2.2A1. For part (d) see LO 1.1C/EK 1.1C3, LO 2.2B/EK 2.2B2. This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, connecting multiple representations, building notational fluency, and communicating.

Sample: 5A
Score: 9

The response earned all 9 points: 1 point in part (a), 2 points in part (b), 3 points in part (c), and 3 points in part (d). In part (a) the response presents a difference quotient and would have earned the point for $\frac{e^{\pi} \cos \pi - e^0 \cos 0}{\pi}$ in line 1 with no simplification. In this case, correct simplification to $\frac{-e^{\pi} - 1}{\pi}$ earned the point. In part (b) the response would have earned the first point for $\frac{df}{dx} = e^x \cos x + e^x(-\sin x)$ in line 1. Responses that go on to factor must do so correctly. The response earned the point with the derivative expression factored correctly to $e^x(\cos x - \sin x)$. The response would have earned the second point for the slope

Question 5 (continued)

$e^{3\pi/2}(0-(-1))$ in line 3 with no simplification. In this case, correct simplification to $e^{3\pi/2}$ earned the point. In part (c) the response earned the first point with $f'(x) = e^x(\cos x - \sin x) = 0$ in line 1. The response earned the second point for identifying $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$ as candidates in line 2 on the right. The response earned the third point for the absolute minimum value of $-\frac{\sqrt{2}}{2}e^{5\pi/4}$ and the global argument appealing to the behavior of $f(x)$ on the entire interval. The candidates test is used to justify that the minimum value is the only negative extreme value on the interval. In part (d) the response earned the first point by stating that $g(x)$ is continuous in line 1 and showing in lines 1 and 3 that $\lim_{x\to\pi/2}(g(x)) = 0$ and $\lim_{x\to\pi/2} f(x) = 0.$ The response earned the second point for a limit attached to a ratio of derivatives $\lim_{x\to\pi/2}\left(\frac{f'(x)}{g'(x)}\right)$ in the last line. The third point was earned for the answer $\frac{-e^{\pi/2}}{2}$.

Sample: 5B
Score: 6

The response earned 6 points: 1 point in part (a), 2 points in part (b), 1 point in part (c), and 2 points in part (d). In part (a) the response presents a difference quotient and would have earned the point for $\frac{-e^{\pi}-1}{\pi}$ in line 1. In this case, the rewriting of the expression as $-\frac{e^{\pi}}{\pi} - \frac{1}{\pi}$ earned the point. In part (b) the response earned the first point for $f'(x) = -e^x \sin x + e^x \cos x.$ The response would have earned the second point for the slope $f'\left(\frac{3\pi}{2}\right) = -e^{3\pi/2}(-1) + e^{3\pi/2}(0)$ in line 2 with no simplification. In this case, correct simplification to $e^{3\pi/2}$ earned the point. In part (c) the response earned the first point with $f'(x) = -e^x \sin x + e^x \cos x = 0.$ Because the response does not identify both $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$ as candidates, the second point was not earned. The third point was not earned because $-\frac{\sqrt{2}e^{-\pi/4}}{2}$ is not the absolute minimum value of f on $0 \le x \le 2\pi.$ Additionally, $x = -\frac{\pi}{4}$ is not in the interval. In part (d) the response did not earn the first point because the conditions that g is continuous and that $\lim_{x\to\pi/2} f(x) = \lim_{x\to\pi/2} g(x) = 0$ are not verified. The response earned the second point in line 1 for a limit attached to a ratio of derivatives $\lim_{x\to\pi/2}\frac{f'(x)}{g'(x)}.$ The third point was earned for the answer $-\frac{e^{\pi/2}}{2}$.

Sample: 5C
Score: 3

The response earned 3 points: 1 point in part (a), 1 point in part (b), 1 point in part (c), and no points in part (d). In part (a) the response presents a difference quotient and would have earned the point for $\frac{e^{\pi}\cos(\pi) - e^{0}\cos(0)}{\pi}$

Question 5 (continued)

in line 1 with no simplification. In this case, correct simplification to $\frac{-e^{\pi} - 1}{\pi}$ earned the point. In part (b) the response earned the first point for $f'(x) = e^x \cos(x) - e^x \sin(x)$ in line 1 on the right. The response would have earned the second point for the slope $f'\left(\frac{3\pi}{2}\right) = e^{3\pi/2} \cos\left(\frac{3\pi}{2}\right) - e^{3\pi/2} \sin\left(\frac{3\pi}{2}\right)$ in line 2 on the right with no simplification. In this case, incorrect simplification to $-e^{3\pi/2}$ did not earn the point. In part (c) the response earned the first point with $f'(x) = e^x \cos(x) - e^x \sin(x) = 0.$ Because the response does not identify both $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$ as candidates, the second point was not earned. The third point was not earned because an absolute minimum value with a global argument is not presented. In part (d) the response did not earn the first point because the conditions that g is continuous and that $\lim_{x \to \pi/2} f(x) = \lim_{x \to \pi/2} g(x) = 0$ are not verified. The second point was not earned because there is no limit attached to a ratio of derivatives. A maximum of 1 point can be earned in part (d) for responses with no limit notation attached to a ratio of derivatives. As a result, the response is not eligible for the third point.

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Sample Student Responses and Scoring Commentary

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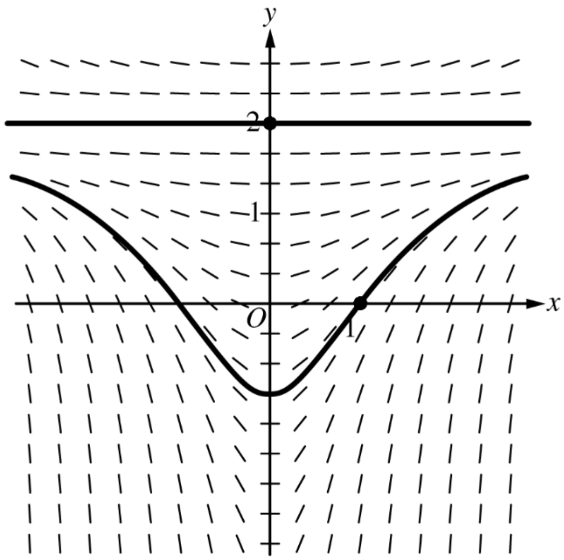
Question 6

(a)



$2 : \begin{cases} 1 : \text{solution curve through } (0, 2) \\ 1 : \text{solution curve through } (1, 0) \end{cases}$

Curves must go through the indicated points, follow the given slope lines, and extend to the boundary of the slope field.

(b) $\left.\dfrac{dy}{dx}\right|_{(x,\, y)=(1,\, 0)} = \dfrac{4}{3}$

An equation for the line tangent to the graph of $y = f(x)$ at $x = 1$ is $y = \frac{4}{3}(x-1)$.

$f(0.7) \approx \frac{4}{3}(0.7 - 1) = -0.4$

$2 : \begin{cases} 1 : \text{equation of tangent line} \\ 1 : \text{approximation} \end{cases}$

(c) $\dfrac{dy}{dx} = \dfrac{1}{3}x(y-2)^2$

$$\int \frac{dy}{(y-2)^2} = \int \frac{1}{3}x\,dx$$

$$\frac{-1}{y-2} = \frac{1}{6}x^2 + C$$

$$\frac{1}{2} = \frac{1}{6} + C \Rightarrow C = \frac{1}{3}$$

$$\frac{-1}{y-2} = \frac{1}{6}x^2 + \frac{1}{3} = \frac{x^2+2}{6}$$

$$y = 2 - \frac{6}{x^2+2}$$

Note: this solution is valid for $-\infty < x < \infty$.

$5 : \begin{cases} 1 : \text{separation of variables} \\ 2 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ \quad\ \text{and uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: $0/5$ if no separation of variables

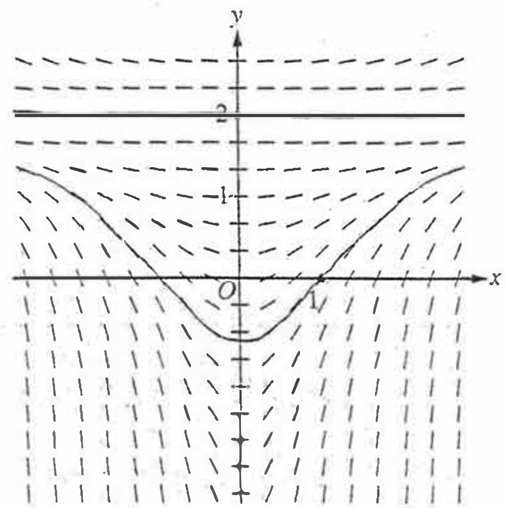
Note: max $3/5$ [1-2-0-0] if no constant of integration

NO CALCULATOR ALLOWED

6A
1of2

6. Consider the differential equation $\frac{dy}{dx} = \frac{1}{3}x(y-2)^2$.

(a) A slope field for the given differential equation is shown below. Sketch the solution curve that passes through the point (0, 2), and sketch the solution curve that passes through the point (1, 0).



(b) Let $y = f(x)$ be the particular solution to the given differential equation with initial condition $f(1) = 0$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 1$. Use your equation to approximate $f(0.7)$.

$y = M_T(x-1) + 0$

$M_T = \frac{1}{3} \cdot 1(0-2)^2$

$= \frac{1}{3}(4)$

$= 4/3$

$y = 4/3(x-1)$

$f(0.7) \approx 4/3(0.7-1)$

$\approx 4/3\left(-\frac{3}{10}\right)$

$\approx -4/10 = -2/5$

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Continue question 6 on page 21.

6A
1 of 2

(c) Find the particular solution $y = f(x)$ to the given differential equation with initial condition $f(1) = 0$.

$u = y - 2$

$du = dy$

$$\int \frac{dy}{(y-2)^2} = \int \frac{1}{3} x\,dx$$

$$\int \frac{du}{u^2} = \frac{1}{3} \int x\,dx$$

$$-\frac{1}{u} = \frac{1}{3}\left(\frac{x^2}{2}\right) + C$$

$$-\frac{1}{(y-2)} = \frac{x^2}{6} + C$$

point: $x = 1$
$y = 0$

$$C = \frac{1}{2} - \frac{1}{6} = \frac{3}{6} - \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

$$\therefore \quad \frac{-1}{y-2} = \frac{x^2}{6} + \frac{1}{3}$$

$$\frac{1}{y-2} = -\frac{x^2}{6} - \frac{1}{3}$$

$$y - 2 = \left(\frac{1}{-\frac{x^2}{\cancel{6}} - \frac{2}{\cancel{6}}}\right)\frac{6}{\cancel{6}}$$

$$y - 2 = \frac{6}{-(x^2+2)}$$

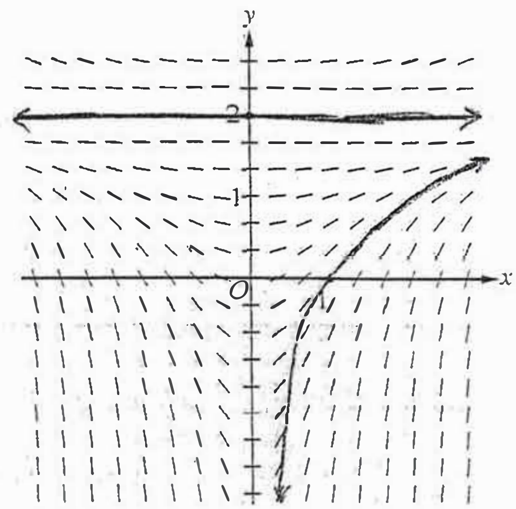
$$y = \frac{6}{-(x^2+2)} + 2$$

NO CALCULATOR ALLOWED

6B
1 of 2

6. Consider the differential equation $\frac{dy}{dx} = \frac{1}{3}x(y-2)^2$.

(a) A slope field for the given differential equation is shown below. Sketch the solution curve that passes through the point $(0, 2)$, and sketch the solution curve that passes through the point $(1, 0)$.



(b) Let $y = f(x)$ be the particular solution to the given differential equation with initial condition $f(1) = 0$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 1$. Use your equation to approximate $f(0.7)$.

$$\frac{dy}{dx} = \frac{1}{3}x(y-2)^2$$

$$= \frac{1}{3}(1)(-2)^2$$

$$= \frac{4}{3}$$

$$\boxed{y = \frac{4}{3}(x-1)}$$

$$y = \frac{4}{3}(.7-1)$$

$$y = \frac{4}{3}(.3)$$

$$= \frac{4}{5}$$

Continue question 6 on page 21.

6B

2 of 2

(c) Find the particular solution $y = f(x)$ to the given differential equation with initial condition $f(1) = 0$.

$$\frac{dy}{dx} = \frac{1}{3}x(y-2)^2$$

$$\int (y-2)^{-2} dy = \int \frac{1}{3}x \, dx$$

$$-(y-2)^{-1} = \frac{1}{6}x^2 + C$$

$$-\frac{1}{y-2} = \frac{1}{6}x^2 + C$$

$$\frac{1}{2} = \frac{1}{6} + C$$

$$\frac{3}{6} - \frac{1}{6} = C$$

$$\frac{1}{3} = C$$

$$\frac{-1}{y-2} = \frac{1}{6}x^2 + \frac{1}{3}$$

$$-1 = \frac{1}{6}x^2 + \frac{1}{3}(y-2)$$

$$-6x^2 - 3 = y - 2$$

$$-6x^2 - 1 = y$$

$$\boxed{y = -6x^2 - 1}$$

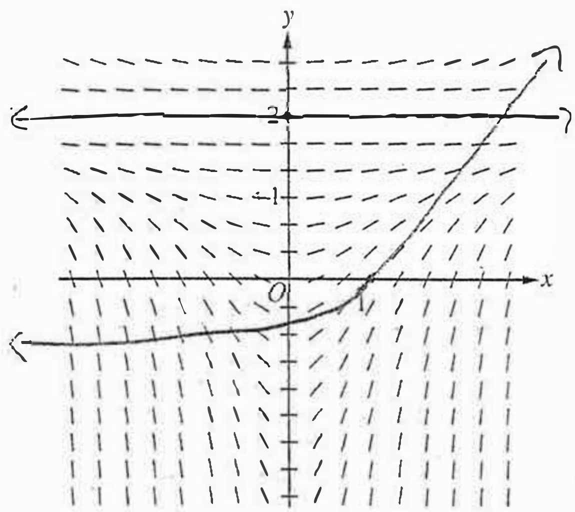
GO ON TO THE NEXT PAGE.

NO CALCULATOR ALLOWED

6C
1 of 2

6. Consider the differential equation $\frac{dy}{dx} = \frac{1}{3}x(y-2)^2$.

(a) A slope field for the given differential equation is shown below. Sketch the solution curve that passes through the point $(0, 2)$, and sketch the solution curve that passes through the point $(1, 0)$.



(b) Let $y = f(x)$ be the particular solution to the given differential equation with initial condition $f(1) = 0$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 1$. Use your equation to approximate $f(0.7)$.

$\frac{dy}{dx} = \frac{1}{3}x(y-2)^2$

$\frac{1}{3}(1)(0-2)^2$

$\frac{1}{3}(-2)^2$

$\frac{1}{3}(4) = \frac{4}{3}$

$\boxed{f(.7) = -2/5}$

$0 = \frac{4}{3}(1) + b$

$0 = \frac{4}{3} + b$

$-4/3$

$-4/3 = b$

$\boxed{y = 4/3x - 4/3}$

$y = 4/3(.7) - 4/3$

$y = \frac{28}{30} - \frac{4}{3}$

$y = \frac{28}{30} - \frac{40}{30}$

$y = -\frac{12}{30}$

$y = -\frac{6}{15}$

$y = -\frac{2}{5}$

Continue question 6 on page 21.

6C
2 of 2

(c) Find the particular solution $y = f(x)$ to the given differential equation with initial condition $f(1) = 0$.

$$\frac{dy}{dx} = \frac{1}{3}x(y-2)^2$$

$$y = \frac{\frac{1}{3}x(y-2)^3}{3} \cdot \frac{y^2}{2} - 2x + 2$$

$U = y-2$

$e = \frac{1}{3}x(U)^2$

$\frac{\frac{1}{3}x(U)^3}{3} \cdot U'$

$\frac{\frac{1}{3}x(y-2)^3}{3} \cdot \frac{y^2}{2} - 2x$

$y = \frac{\frac{1}{3}x(y-2)^3}{3} \cdot \frac{y^2}{2} - 2x + C$

$0 = \frac{\frac{1}{3}(1)(0-2)^3}{3} \cdot \frac{0}{2} - 2(1) + C$

$0 = \frac{\frac{1}{3}(-8)}{3} \cdot 0 - 2$

$0 = -2 + C$

$+2 \quad +2$

$C = 2$

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2018 SCORING COMMENTARY

Question 6

Overview

This problem deals with the differential equation $\frac{dy}{dx} = \frac{1}{3}x(y-2)^2$. In part (a) students were given a slope field for the differential equation and asked to sketch solution curves corresponding to solutions that pass through the points $(0, 2)$ and $(1, 0)$. A correct response should be two sketched curves that pass through the indicated points, follow the given slope lines, and extend to the boundaries of the provided slope field. In part (b) students were given that a solution f satisfies $f(1) = 0$ and asked to supply an equation for the line tangent to the graph of f at $x = 1$. Students were then to use this equation to approximate $f(0.7)$. A correct response should use the fact that the slope of the tangent line is the value of the derivative of f at the indicated point, and this value can be computed from substitution of $(x, y) = (1, 0)$ in the differential equation. Combining the slope and the point $(1, 0)$ gives the tangent line equation $y = \frac{4}{3}(x-1)$; substituting $x = 0.7$ into this equation gives the requested approximation for $f(0.7)$. In part (c) students were asked to find the particular solution $y = f(x)$ to the given differential equation that satisfies $f(1) = 0$. A correct response should employ the method of separation of variables and use the initial condition $f(1) = 0$ to resolve the constant of integration to arrive at the solution $f(x) = 2 - \frac{6}{x^2+2}$.

For part (a) see LO 2.3F/EK 2.3F1. For part (b) see LO 2.3B/EK 2.3B2. For part (c) see LO 3.3B(a)/EK 3.3B5, LO 3.5A/EK 3.5A2. This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, connecting multiple representations, building notational fluency, and communicating.

Sample: 6A
Score: 9

The response earned all 9 points: 2 points in part (a), 2 points in part (b), and 5 points in part (c). In part (a) the response earned the first point with the solution curve $y = 2$ drawn through the point $(0, 2)$. The response earned the second point with the solution curve through $(1, 0)$ that follows the slope lines and extends to both boundaries of the slope field. In part (b) the response earned the first point with the tangent line equation $y = \frac{4}{3}(x-1)$. The response would have earned the second point for $f(0.7) \approx \frac{4}{3}(0.7-1)$ with no simplification. In this case, correct simplification to $-\frac{2}{5}$ earned the point. In part (c) the response earned the first point with a correct separation of variables in line 1, $\int \frac{dy}{(y-2)^2} = \int \frac{1}{3}x\,dx$. The response earned the second and third points with the correct antiderivatives $-\frac{1}{(y-2)} = \frac{x^2}{6} + C$ in line 4. The response earned the fourth point by both including C with the antiderivatives in line 4 and for substituting 1 for x and 0 for y in line 5. The response earned the fifth point by presenting a correct solution, $y = \frac{6}{-\left(x^2+2\right)} + 2$.

Question 6 (continued)

Sample: 6B
Score: 6

The response earned 6 points: 1 point in part (a), 1 point in part (b), and 4 points in part (c). In part (a) the response earned the first point with the solution curve $y = 2$ drawn through the point $(0, 2)$. The response did not earn the second point because the curve through $(1, 0)$ does not follow the slope lines correctly from the left boundary to the right boundary of the slope field. In part (b) the response earned the first point with the tangent line equation $y = \frac{4}{3}(x - 1)$ in line 1 on the right. The response would have earned the second point for the approximation $y = \frac{4}{3}(.7 - 1)$ in line 2 on the right with no simplification. In this case, incorrect simplification to $\frac{4}{9}$ did not earn the second point. In part (c) the response earned the first point with a correct separation of variables in line 2, $\int (y - 2)^{-2}\, dy = \int \frac{1}{3}x\, dx.$ The response earned the second and third points with the correct antiderivatives $-(y - 2)^{-1} = \frac{1}{6}x^2 + C$ in line 3. The response earned the fourth point with both the constant of integration in line 3 and for substituting 1 for x and 0 for y resulting in $\frac{1}{2} = \frac{1}{6} + C$ in line 5. The response does not correctly solve for $y = f(x),$ so the fifth point was not earned.

Sample: 6C
Score: 3

The response earned 3 points: 1 point in part (a), 2 points in part (b), and no points in part (c). In part (a) the response earned the first point with the solution curve $y = 2$ drawn through the point $(0, 2)$. The response did not earn the second point because the curve through $(1, 0)$ does not follow the slope lines correctly. In part (b) the response earned the first point with the tangent line $y = \frac{4}{3}x - \frac{4}{3}$ in the box on the right. The response would have earned the second point for $y = \frac{4}{3}(.7) - \frac{4}{3}$ below the box on the right with no simplification. In this case, correct simplification of the approximation to $f(.7) = -\frac{2}{5}$ earned the second point. In part (c) the response does not separate the variables. The response is not eligible for any points in part (c).